

ARCHIVUM MATHEMATICUM (BRNO)

Tomus 41 (2005), 151 – 155

ASYMPTOTIC STABILITY FOR SETS OF POLYNOMIALS

THOMAS W. MÜLLER AND JAN-CHRISTOPH SCHLAGE-PUCHTA

ABSTRACT. We introduce the concept of asymptotic stability for a set of complex functions analytic around the origin, implicitly contained in an earlier paper of the first mentioned author (“Finite group actions and asymptotic expansion of $e^{P(z)^n}$ ”, *Combinatorica* 17 (1997), 523 – 554). As a consequence of our main result we find that the collection of entire functions $\exp(\mathfrak{P})$ with \mathfrak{P} the set of all real polynomials $P(z)$ satisfying Hayman’s condition $[z^n]\exp(P(z)) > 0$ ($n \geq n_0$) is asymptotically stable. This answers a question raised in loc. cit.

1. ASYMPTOTIC STABILITY

Let \mathfrak{F} be a set of complex functions analytic in the origin, and for $f \in \mathfrak{F}$ let $f(z) = \sum_n \alpha_n^f z^n$ be the expansion of f around 0. \mathfrak{F} is termed *asymptotically stable*, if

- (i) $\forall f \in \mathfrak{F} \exists n_f \in \mathbb{N}_0 \forall n \geq n_f : \alpha_n^f \neq 0$,
- (ii) $\forall f, g \in \mathfrak{F} : \alpha_n^f \sim \alpha_n^g \rightarrow f = g$ in a neighbourhood of 0.

Here, for arithmetic functions f and g , the notation $f(n) \sim g(n)$ is short for

$$g(n) = f(n)(1 + o(1)), \quad n \rightarrow \infty.$$

A set of polynomials $\mathfrak{P} \subseteq \mathbb{C}[z]$ is called asymptotically stable, if the set of entire functions

$$\mathfrak{F} = \exp(\mathfrak{P}) := \{e^{P(z)} : P(z) \in \mathfrak{P}\}$$

is asymptotically stable. Define the degree of the zero polynomial to be -1 . For a polynomial $P(z) = \sum_{\delta=0}^d c_\delta z^\delta$ of exact degree $d \geq 1$ with real coefficients c_δ consider the following two conditions:

- (\mathcal{G}) $c_\delta = 0$ for $d/2 < \delta < d$,
- (\mathcal{H}) $[z^n]e^{P(z)} > 0$ for all sufficiently large n .

1991 *Mathematics Subject Classification*: 30B10.

Key words and phrases: power series, coefficients, asymptotic expansion.

Received May 21, 2003, revised August 2004.

Here, $[z^n]f(z)$ denotes the coefficient of z^n in the expansion of $f(z)$ around the origin. Asymptotically stable sets of functions first appeared in [3], where it was shown among other things that the set of polynomials

$$\mathfrak{P}_0 = \{P(z) \in \mathbb{R}[z] : P(z) \text{ satisfies } (\mathcal{G}) \text{ and } (\mathcal{H})\}$$

is asymptotically stable. Since for a finite group G we have¹

$$\sum_{n=0}^{\infty} |\text{Hom}(G, S_n)| \frac{z^n}{n!} = \exp \left(\sum_{\nu} |\{U : (G : U) = \nu\}| \frac{z^\nu}{\nu} \right),$$

asymptotic stability of \mathfrak{P}_0 implies in particular the following curious phenomenon (“asymptotic stability” of finite groups):

If for two finite groups G and H we have $|\text{Hom}(G, S_n)| \sim |\text{Hom}(H, S_n)|$ as $n \rightarrow \infty$, then these arithmetic functions must in fact coincide.

Condition (\mathcal{H}) arises in the work of Hayman [2], where it is shown that for a real polynomial $P(z)$ of degree at least 1 the function $e^{P(z)}$ is admissible in the complex plane in the sense of [2, pp. 68 - 69] if and only if (\mathcal{H}) holds; cf. [2, Theorem X]. The gap condition (\mathcal{G}) has turned out to be an efficient way of exploiting the fact that polynomials $P(z)$ arising from enumerative problems very often have the property that

$$\text{supp}(P(z)) \subseteq \{\delta : \delta \mid \deg(P(z))\}.$$

In [3] the question was raised whether condition (\mathcal{G}) could be dropped while still maintaining asymptotic stability, i.e., whether the larger set of polynomials

$$(1) \quad \mathfrak{P} = \{P(z) \in \mathbb{R}[z] : P(z) \text{ satisfies } (\mathcal{H})\}$$

is asymptotically stable. The purpose of this note is to establish the following result, which in particular provides an affirmative answer to the latter question.

Theorem. *Let $P_1(z), P_2(z) \in \mathbb{R}[z]$ satisfy Hayman’s condition (\mathcal{H}) , for $i = 1, 2$ let $\{\alpha_n^{(i)}\}_{n \geq 0}$ be the coefficients of $e^{P_i(z)}$, and put $\Delta(z) := P_1(z) - P_2(z)$ as well as $m := \max(\deg(P_1(z)), \deg(P_2(z)))$.*

(i) *Suppose that either $0 \leq \mu < m$, or $\mu = m$ and $\deg(P_1(z)) = \deg(P_2(z))$. Then we have $\deg(\Delta(z)) = \mu$ if and only if $|\log \alpha_n^{(1)} - \log \alpha_n^{(2)}| \asymp n^{\mu/m}$.*

(ii) *If $\deg(P_1(z)) \neq \deg(P_2(z))$, then $|\log \alpha_n^{(1)} - \log \alpha_n^{(2)}| \asymp n \log n$.*

Here, $f(n) \asymp g(n)$ means that $f(n)$ and $g(n)$ are of the same order of magnitude; that is, there exist positive constants c_1, c_2 such that $c_1 f(n) \leq g(n) \leq c_2 f(n)$ for all n .

Corollary. *The set of polynomials \mathfrak{P} defined in (1) is asymptotically stable.*

Proof. If $P_1(z), P_2(z) \in \mathbb{R}[z]$ are polynomials satisfying condition (\mathcal{H}) as well as $\alpha_n^{(1)} \sim \alpha_n^{(2)}$, then $\log \alpha_n^{(1)} - \log \alpha_n^{(2)} = o(1)$. By our theorem, $\deg(\Delta(z)) \notin [0, m]$, and hence $P_1(z) = P_2(z)$. \square

¹Cf. for instance [1, Prop. 1] or [4, Exercise 5.13].

2. PROOF OF THE THEOREM

For $i = 1, 2$ put $P_i(z) = \sum_{\delta=0}^{d_i} c_{d_i}^{(i)} z^\delta$ with $c_{d_i}^{(i)} \neq 0$. Our assumptions that $P_1(z)$ and $P_2(z)$ have real coefficients and satisfy (\mathcal{H}) ensure via [2, Theorem X] that the functions $\exp(P_i(z))$ are admissible in the complex plane; in particular, in view of [2, formula (1.2)], we have $c_{d_i}^{(i)} > 0$. By [2, Theorem I] we find that, for $i = 1, 2$,

$$\alpha_n^{(i)} \sim \frac{\exp(P_i(\vartheta_n^{(i)}))}{(\vartheta_n^{(i)})^n \sqrt{2\pi b_i(\vartheta_n^{(i)})}} \quad (n \rightarrow \infty),$$

where $\vartheta_n^{(i)}$ is the positive real root of the equation $\vartheta P_i'(\vartheta) = n$, and $b_i(\vartheta) = \vartheta P_i'(\vartheta) + \vartheta^2 P_i''(\vartheta)$. Since $c_{d_i}^{(i)} > 0$, the root $\vartheta_n^{(i)}$ is well defined and increasing for sufficiently large n , and unbounded as $n \rightarrow \infty$. This gives $\vartheta_n^{(i)} \sim \left(\frac{n}{d_i c_{d_i}^{(i)}}\right)^{1/d_i}$ and $b_i(\vartheta_n^{(i)}) \sim d_i n$, and hence

$$(2) \quad \alpha_n^{(i)} \sim \frac{\exp(P_i(\vartheta_n^{(i)}))}{(\vartheta_n^{(i)})^n \sqrt{2\pi d_i n}} \quad (n \rightarrow \infty).$$

Formula (2) implies that

$$(3) \quad \begin{aligned} \log \alpha_n^{(1)} - \log \alpha_n^{(2)} &= P_1(\vartheta_n^{(1)}) - P_2(\vartheta_n^{(2)}) - n(\log \vartheta_n^{(1)} - \log \vartheta_n^{(2)}) \\ &\quad - \frac{1}{2}(\log d_1 - \log d_2) + o(1). \end{aligned}$$

First consider case (ii), that is, the case when $d_1 \neq d_2$. Then, by (3),

$$\log \alpha_n^{(1)} - \log \alpha_n^{(2)} = \left(\frac{1}{d_2} - \frac{1}{d_1}\right) n \log n + \mathcal{O}(n),$$

that is,

$$|\log \alpha_n^{(1)} - \log \alpha_n^{(2)}| \asymp n \log n$$

as claimed.² Next suppose that $d_1 = d_2$. Then the right-hand side of (3) becomes

$$d_1^{-1} \log(c_{d_1}^{(1)}/c_{d_2}^{(2)}) n + o(n);$$

in particular, we have $\deg(\Delta(z)) = m$ if and only if $|\log \alpha_n^{(1)} - \log \alpha_n^{(2)}| \asymp n$, which proves the last part of (i). Thirdly, for $m = 1$,

$$\log \alpha_n^{(1)} - \log \alpha_n^{(2)} = c_0^{(1)} - c_0^{(2)} + n \log(c_1^{(1)}/c_1^{(2)}) + o(1),$$

in particular, $\deg(\Delta(z)) = 0$ if and only if $|\log \alpha_n^{(1)} - \log \alpha_n^{(2)}| \asymp 1$. Hence, we may assume for the remainder of the argument that $m \geq 2$.

²Here, as well as in certain other places below, a more precise estimate than the one stated is obtained, but not needed in the argument.

Now suppose that $0 \leq \mu := \deg(\Delta(z)) < m$. We want to show that in this case $|\log \alpha_n^{(1)} - \log \alpha_n^{(2)}| \asymp n^{\mu/m}$. We have

$$\begin{aligned}
 (4) \quad n - \vartheta_n^{(1)} P_2'(\vartheta_n^{(1)}) &= \vartheta_n^{(1)} [P_1'(\vartheta_n^{(1)}) - P_2'(\vartheta_n^{(1)})] \\
 &= \vartheta_n^{(1)} \Delta'(\vartheta_n^{(1)}) \\
 &= a\mu(\vartheta_n^{(1)})^\mu + o(n^{\mu/m}),
 \end{aligned}$$

where a is the leading coefficient of $\Delta(z)$, which we may suppose without loss of generality to be positive. Expanding $\vartheta P_2'(\vartheta)$ as Taylor series around $\vartheta_n^{(1)}$, we find that

$$\begin{aligned}
 (5) \quad \vartheta P_2'(\vartheta) - \vartheta_n^{(1)} P_2'(\vartheta_n^{(1)}) &= \left(c_m^{(2)} m^2 (\vartheta_n^{(1)})^{m-1} + \mathcal{O}(n^{\frac{m-2}{m}}) \right) (\vartheta - \vartheta_n^{(1)}) \\
 &\quad + \mathcal{O}\left(n^{\frac{m-2}{m}} (\vartheta - \vartheta_n^{(1)})^2 + (\vartheta - \vartheta_n^{(1)})^m \right).
 \end{aligned}$$

If ϑ runs through the interval

$$I = \left[\vartheta_n^{(1)} - \frac{2a\mu}{m^2 c_m^{(1)}}, \vartheta_n^{(1)} + \frac{2a\mu}{m^2 c_m^{(1)}} \right],$$

the right-hand side of (5) covers a range containing the interval

$$\left[-(2 - \varepsilon)a\mu(\vartheta_n^{(1)})^{m-1}, (2 - \varepsilon)a\mu(\vartheta_n^{(1)})^{m-1} \right]$$

for every given $\varepsilon > 0$ and sufficiently large n depending on ε . Combining this observation with (4), we find that $n - \vartheta P_2'(\vartheta)$ changes sign in I , that is, $\vartheta_n^{(2)} \in I$ for large n ; in particular we have $\vartheta_n^{(2)} - \vartheta_n^{(1)} = \mathcal{O}(1)$. Since $m \geq 2$, setting $\vartheta = \vartheta_n^{(2)}$ in (5) and rewriting the left-hand side via (4) now gives

$$(6) \quad a\mu(\vartheta_n^{(1)})^\mu = \left(c_m^{(1)} m^2 (\vartheta_n^{(1)})^{m-1} + \mathcal{O}(n^{\frac{m-2}{m}}) \right) (\vartheta_n^{(2)} - \vartheta_n^{(1)}) + o(n^{\mu/m}).$$

For x, y real, $x \rightarrow \infty$, and $x - y = \mathcal{O}(1)$,

$$P_2(x) - P_2(y) = (x - y) P_2'(x) + \mathcal{O}((x - y) x^{m-2}).$$

Hence, applying (6), we have as $n \rightarrow \infty$

$$\begin{aligned}
 P_1(\vartheta_n^{(1)}) - P_2(\vartheta_n^{(2)}) &= \Delta(\vartheta_n^{(1)}) + P_2(\vartheta_n^{(1)}) - P_2(\vartheta_n^{(2)}) \\
 &= \Delta(\vartheta_n^{(1)}) + (\vartheta_n^{(1)} - \vartheta_n^{(2)}) P_2'(\vartheta_n^{(1)}) + \mathcal{O}((\vartheta_n^{(1)} - \vartheta_n^{(2)})(\vartheta_n^{(1)})^{m-2}) \\
 &= a \left(1 - \frac{\mu}{m} \right) \left(\frac{n}{m c_m^{(1)}} \right)^{\mu/m} + o(n^{\mu/m}).
 \end{aligned}$$

Moreover, using (6) again,

$$\begin{aligned} \log \vartheta_n^{(2)} - \log \vartheta_n^{(1)} &= \log \left(1 + \frac{\vartheta_n^{(2)} - \vartheta_n^{(1)}}{\vartheta_n^{(1)}} \right) \\ &= \frac{\vartheta_n^{(2)} - \vartheta_n^{(1)}}{\vartheta_n^{(1)}} + o\left(\frac{\vartheta_n^{(2)} - \vartheta_n^{(1)}}{\vartheta_n^{(1)}}\right) \\ &= \frac{a\mu}{m} n^{-1} \left(\frac{n}{m c_m^{(1)}} \right)^{\mu/m} + o(n^{\frac{\mu-m}{m}}). \end{aligned}$$

Inserting these estimates in (3) now yields

$$\begin{aligned} \log \alpha_n^{(1)} - \log \alpha_n^{(2)} &= a \left(1 - \frac{\mu}{m} \right) \left(\frac{n}{m c_m^{(1)}} \right)^{\mu/m} + \frac{a\mu}{m} \left(\frac{n}{m c_m^{(1)}} \right)^{\mu/m} + o(n^{\mu/m}) \\ &= a \left(\frac{n}{m c_m^{(1)}} \right)^{\mu/m} + o(n^{\mu/m}) \asymp n^{\mu/m}, \end{aligned}$$

and our theorem is proven.

REFERENCES

- [1] Dress, A. and Müller, T., *Decomposable functors and the exponential principle*, Adv. in Math. **129** (1997), 188–221.
- [2] Hayman, W., *A generalisation of Stirling's formula*, J. Reine u. Angew. Math. **196** (1956), 67–95.
- [3] Müller, T., *Finite group actions and asymptotic expansion of $e^{P(z)}$* , Combinatorica **17** (1997), 523–554.
- [4] Stanley, R. P., *Enumerative Combinatorics* vol. 2, Cambridge University Press, 1999.

THOMAS W. MÜLLER

SCHOOL OF MATHEMATICAL SCIENCES, QUEEN MARY AND WESTFIELD COLLEGE
MILE END ROAD, LONDON E1 4NS, ENGLAND
E-mail: T.W.Muller@qmul.ac.uk

JAN-CHRISTOPH SCHLAGE-PUCHTA

MATHEMATISCHES INSTITUT, UNIVERSITÄT FREIBURG
ECKERSTRASSE 1, 79104 FREIBURG, GERMANY
E-mail: jcp@mathematik.uni-freiburg.de